Constraint Programming Illustrated on the Subgraph Isomorphism Problem

Christine Solnon
CITI, INSA Lyon / Inria

14th IAPR-TC15 Workshop on Graph-based Representations in Pattern Recognition

What is Constraint Programming (CP)?

"Constraint programming represents one of the closest approaches computer science has yet made to the Holy Grail of programming: the user states the problem, the computer solves it."

Eugene C. Freuder

In other words: CP = Model + Search

- Model $\rightsquigarrow$ Define a mathematical model by means of variables and constraints
- Search $\rightsquigarrow$ Use a generic solver to search for a solution

What is the difference with ILP (Integer Linear Programming) or SAT?

- Richer modelling language
  - ILP: Model = Linear inequations + Linear objective function
  - SAT: Model = Boolean formula
  - CP: Model = Conjunction of various kinds of constraints
    $\rightsquigarrow$ Each constraint comes with its own propagation algorithm
    $\rightsquigarrow$ If your favorite constraint does not exist, you can create it!
- Different solving approach: Branch & Propagate (vs Branch & Bound/Cut/Price for ILP or CDCL for SAT)

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Overview of the talk

1. **Modelling with CP**
2. Generic CP Solving Algorithms
3. LAD2025: A Constraint-based Solver for the SIP
4. Conclusion

Modelling with CP

CP model = $(X, D, C)[+F]$

- X = Set of variables (unknowns)
- For each variable $x_i \in X$, $D(x_i)$ = domain of x_i
  $\rightsquigarrow$ Set of values that may be assigned to x_i
- C = Constraints (relations between variables of X)
- [Optionally] $F : X \to \mathbb{R}$ = objective function to optimize

Solution of a CP model:

Assignment of a value to every variable of X such that:

- Each variable $x_i \in X$ is assigned to a value that belongs to $D(x_i)$
- Every constraint of C is satisfied
- [Optionally] F is maximized (or minimized)

Remark: A problem may have several different models...

Example: Subgraph Isomorphism Problem (SIP)

Definition of the SIP:

Given $G_p = (V_p, E_p)$ and $G_t = (V_t, E_t)$:
find an injective mapping $f : V_p \to V_t$ such that
$\forall (i,j) \in E_p, (f(i), f(j)) \in E_t$

Example of SIP instance:

$G_p = (V_p, E_p)$ $G_t = (V_t, E_t)$

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CP model introduced by Ullmann in 1976[1]:

- Variables: $X = \{x_i | i \in V_p\}$
- Domains:
  $\forall i \in V_p, D(x_i) = \{u \in V_t : d^\circ(i) \leq d^\circ(u)\}$
- Constraints:
  - f must be injective: $\forall \{i, j\} \subseteq V_p, x_i \neq x_j$
  - Edge constraints:
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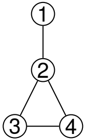

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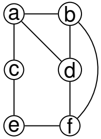

$G_t = (V_t, E_t)$

Domains:

- $D(x_1) = D(x_3) = D(x_4) = \{a, b, c, d, e, f\}$
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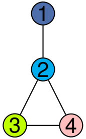

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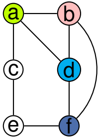

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Example of solution:

$x_1 = f$, $x_2 = d$, $x_3 = a$, $x_4 = b$

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Other CP model with binary variables:

- Variables: $X = \{x_{ij} | i \in V_p, j \in V_t\}$
- Domains: $\forall i \in V_p, \forall j \in V_t, D(x_{ij}) = \{0, 1\}$
- Constraints:
  - f must be injective:
    $\forall i \in V_p, \sum_j x_{ij} = 1 \wedge \forall j \in V_t, \sum_i x_{ij} \leq 1$
  - Edge constraints:
    $\forall (i_1, i_2) \in E_p, \forall (j_1, j_2) \in V_t \times V_t \setminus E_t, x_{i_1 j_1} + x_{i_2 j_2} < 2$

Example of SIP instance:

$G_p = (V_p, E_p)$ $\quad$ $G_t = (V_t, E_t)$

Example of solution:

- $x_{1f} = x_{2d} = x_{3a} = x_{4b} = 1$
- All other variables are set to 0

1 Ullmann: *An algorithm for subgraph isomorphism*, in J. ACM, 1976

CP Languages and Libraries

First CP language introduced by Jean-Louis Laurière in 1976:

ALICE

Extensions of Prolog:

CHIP, Prolog V, Gnu-Prolog, Sicstus Prolog, Picat, ...

Libraries:

- Open source: Choco (Java), PyCSP3 (Python), OR-Tools/Google (C++, Java, Python, ...), etc
- Commercial: CP-Optimizer (IBM)

Modelling languages (accepted by most solvers):

MiniZinc, XCSP3

Example: SIP in PyCSP3 (using a notebook)

Define the pattern graph and the target graph

```
[3]  n_p = 4; n_t = 6
     e_p = [(0,1), (0,3), (1,2), (1,3), (2,3)]
     e_t = [(0,1), (0,4), (1,0), (1,2), (1,3), (2,1), (2,3), (2,5), (3,1), (3,2), (3,4), (4,0), (4,3), (5,1), (5,2)]
     deg_p = [2, 3, 2, 3]
     deg_t = [2, 4, 3, 3, 2, 2]
```

Declare variables and domains

```
[4]  x = VarArray(size=n_p, dom=range(n_t))
     for i in range(n_p) :
       x[i] in {j for j in range(n_t) if deg_p[i] <= deg_t[j]}
```

Declare constraints... and solve!

```
[6]  satisfy(AllDifferent(x),
             [(x[i], x[j]) in e_t for (i, j) in e_p])
     if solve() is SAT:
         print(values(x))
```

```
[5, 1, 3, 2]
```

OK, the model is nice (isn't it?)... but is it efficient?

Comparison of ACE[1] (default solver of PyCSP3) with RI[2] and LAD2025[3] on a subset of instances

Number of instances NOT solved within 100s:

	LV	rand	meshes	all
ACE default	85	30	46	161
ACE tuned	72	22	37	131
RI	152	8	284	324
LAD2025	32	5	1	38

- ACE solves more instances when tuning parameters
- ACE solves more instances than RI on LV and meshes
  But it is less successful on rand instances
- ACE is outperformed by LAD2025

1 Lecoutre: *ACE, a generic constraint solver*, arXiv, 2023

2 Bonnici, Giugno. *On the variable ordering in subgraph isomorphism algorithms*, in IEEE Trans. Bioinform., 2017

3 Solnon: *LAD2025, a Constraint-Based Solver for the Subgraph Isomorphism Problem*, 2025 (submitted)

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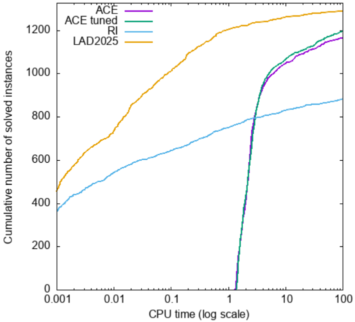

Conclusion:

- ACE solves less instances than dedicated approaches when the time limit is smaller than 3s
- But it solves more instances than RI when the time limit is larger than 3s
- Easy to add application-specific constraints or objective functions

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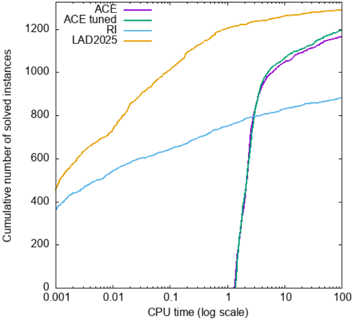

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- **Easy to add application-specific constraints or objective functions**

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Branch & Propagate Solving Approach

Branch:

- Choose an unassigned variable x_i
- For each $v \in D(x_i)$, recursively solve a subproblem where v is assigned to x_i

$\leadsto$ Stop when a solution is found; Backtrack when an inconsistency is detected

Propagate constraints at each subproblem:

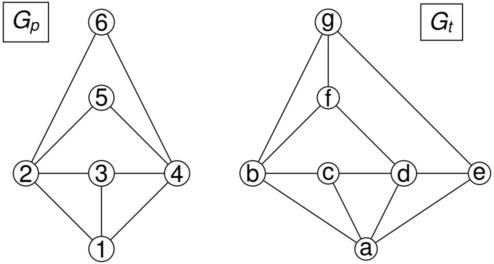
- Goal: Remove inconsistent values from variable domains
  $\leadsto$ Inconsistency detected whenever a domain is wiped out
- Each constraint comes with its own propagation algorithms
  $\leadsto$ Easy to add new kinds of constraints

A first (very simple) propagation: Forward Checking (FC)

Propagate a constraint whenever all its variables but one are assigned

$\rightsquigarrow$ Remove inconsistent values from the domains of unassigned variables

Example of SIP instance:



Initial domains:

- $D(x_1) = D(x_3) = D(x_5) = D(x_6) = \{a, b, c, d, e, f, g\}$
- $D(x_2) = D(x_4) = \{a, b, d\}$

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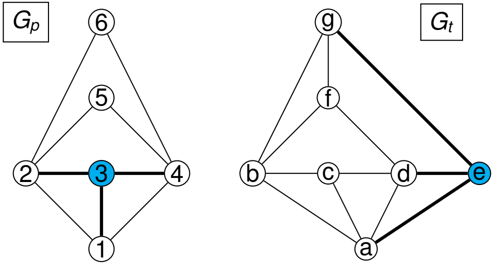

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FC propagation when $x_3 = e$:

- $\forall i \in \{1, 5, 6\}, x_i \neq e$
  $\rightsquigarrow$ Remove e from $D(x_i)$
- $\forall i \in \{1, 2, 4\}, (x_i, e) \in E_t$
  $\rightsquigarrow$ Remove b, c, and f from $D(x_1)$
  $\rightsquigarrow$ Remove b from $D(x_2)$ and $D(x_4)$

This is similar to what is done in VF2[1]

[1] Cordella, Foggia, Sansone, Vento. *A (sub)graph isomorphism algorithm for matching large graphs*, in PAMI, 2004

Stronger propagation: Ensuring Domain Consistency

A constraint c defined over a set $S \subseteq X$ of k variables is Domain Consistent (DC) if:

$\forall x_i \in S, \forall v_i \in D(x_i), \forall x_j \in S \setminus \{x_i\}, \exists v_j \in D(x_j)$ such that c is satisfied by the assignment $x_1 = v_1, ..., x_k = v_k$

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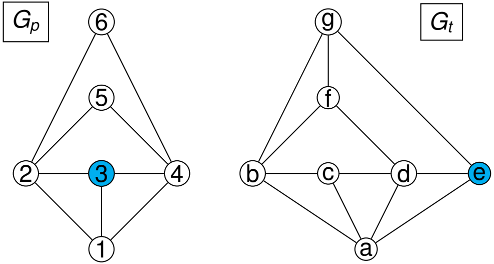

Let's assume that $x_3 = e$.

Domains reduced by FC in this case:

- $D(x_1) = \{a, d, g\}$
- $D(x_2) = \{a, d\}$
- $D(x_4) = \{a, d\}$
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Can we do better?

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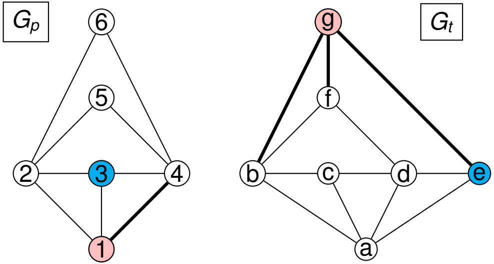

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Can we do better?

DC propagation of the constraint "$(x_1, x_4) \in E_t$":

If $x_1 = g$ then x_4 cannot be assigned to a neighbour of g

$\rightsquigarrow$ Remove g from $D(x_1)$

This corresponds to what is done by Ullmann[1]

[1] Ullmann: *An algorithm for subgraph isomorphism*, in J. ACM, 1976

Historical Notes on DC (aka as Arc Consistency)

- DC has been introduced by Ullmann in 1976 for solving the SIP[1]
- AC3 is the first "efficient" algorithm to ensure DC for binary constraints
  - Introduced by Mackworth in 1977[2]
  - Time complexity in $\mathcal{O}(ed^3)$ where e = number of constraints and d = maximum domain size
  - Space complexity in $\mathcal{O}(e)$
- AC4 is introduced by Mohr & Henderson in 1986[3]
  - Space and time complexities in $\mathcal{O}(ed^2)$
- Many (really many) algorithms introduced since then, with different time and space tradeoffs
  $\rightsquigarrow$ See the last volume of TAOCP[4]

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1 Ullmann. *An algorithm for subgraph isomorphism*, in J. ACM, 1976
2 Mackworth. *Consistency in networks of relations*, in Artificial Intelligence, 1977
3 Mohr & Henderson. *Arc and path consistency revisited*, in Artificial Intelligence, 1986
4 Knuth. *The Art of Computer Programming. Section 7.2.2.3: Constraint Satisfaction*, 2025

Global constraints

What is a global constraint?

Constraint defined over a set of variables (the cardinality of which is not fixed)

Examples of global constraints:

- allDifferent$(x_1, \ldots, x_n) \Leftrightarrow \forall \{x_i, x_j\} \subseteq \{x_1, \ldots, x_n\}, x_i \neq x_j$
- sum$(x_1, \ldots, x_n, s) \Leftrightarrow \sum_{i=1}^{n} x_i = s$
- atLeast$(x_1, \ldots, x_n, k, v) \Leftrightarrow (\sum_{i=1}^{n} [[x_i = v]]) \geq k$ where $[[x_i = v]] = 1$ if $x_i = v$, and 0 otherwise
- ... and many others (see our catalog of 423 global constraints[1])

$\leadsto$ Ease the modelling step

How to propagate a global constraint?

- First possibility: Decompose it into existing constraints and use existing propagators
- Second possibility: Design a dedicated propagator

[1] https://sofdem.github.io/gccat/

Example: Propagation of allDifferent[1]

Suppose that:

- $D(x_1) = \{a, b, d\}$
- $D(x_2) = D(x_3) = \{b, c\}$
- $D(x_4) = \{c, d, e\}$

Propagation of allDifferent(x_1, x_2, x_3, x_4)?

$\rightsquigarrow$ Remove values that cannot belong to a solution

1 Régin. *A filtering algorithm for constraints of difference in CSPs*, in AAAI 1994

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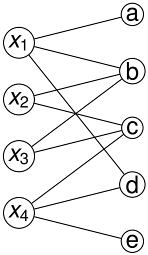

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Algorithm for propagating allDifferent:

- Build a variable/value bipartite graph
- Add a source r and a sink s
- Search for a maximum flow from r to s
  - $\rightsquigarrow$ Algorithm of Hopcroft & Karp[2] in $\mathcal{O}(p^{5/2})$
  - $\rightsquigarrow$ Inconsistency if $|f| < |X|$
- Build the "final flow graph" and search for SCC
  - $\rightsquigarrow$ Algorithm of Tarjan[3] in $\mathcal{O}(p)$
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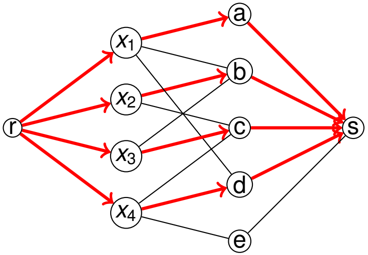

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3 Tarjan. *Depth-first search and linear graph algorithms*, in SIAM J. Comput., 1972

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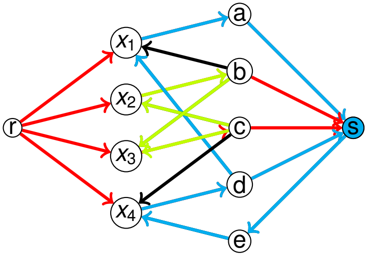

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4 Berge. *Graphs and Hypergraphs*, 1973

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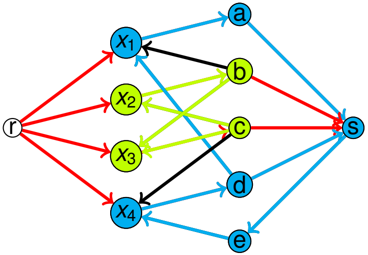

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  - $\rightsquigarrow$ Inconsistency if $|f| < |X|$
- Build the "final flow graph" and search for SCC
  - $\rightsquigarrow$ Algorithm of Tarjan[3] in $\mathcal{O}(p)$
- Remove i from $D(x_j)$ if SCC$(i) \neq$SCC(x_j)[3]
  - $\rightsquigarrow$ Remove b from $D(x_1)$ and c from $D(x_4)$

---

1 Régin. *A filtering algorithm for constraints of difference in CSPs*, in AAAI 1994

2 Hopcroft & Karp. *An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs*, in SIAM J. Comput., 1973

3 Tarjan. *Depth-first search and linear graph algorithms*, in SIAM J. Comput., 1972

4 Berge. *Graphs and Hypergraphs*, 1973

Ordering heuristics

Branch (recall):

- Choose an unassigned variable x_i
- For each $v \in D(x_i)$, recursively solve a subproblem where v is assigned to x_i

$\rightsquigarrow$ Stop when a solution is found; Backtrack when an inconsistency is detected

Classical variable ordering heuristics:

- *deg*: Variable involved in the largest number of constraints $\rightsquigarrow$ Reduce tree depth
- *dom*: Variable with the smallest domain $\rightsquigarrow$ Reduce tree width
- $\frac{dom}{deg}$: Compromise between *dom* and *deg*
- $\frac{dom}{wdeg}$: Each constraint has a weight (incremented on failures)
  $\rightsquigarrow$ divide $|D(x_i)|$ by sum of weights of constraints associated with x_i[1]

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1 Boussemart et al. *Boosting systematic search by weighting constraints*, in ECAI 2004

2

Ordering heuristics

Branch (recall):

- Choose an unassigned variable x_i
- For each $v \in D(x_i)$, recursively solve a subproblem where v is assigned to x_i

$\rightsquigarrow$ Stop when a solution is found; Backtrack when an inconsistency is detected

Value ordering heuristics:

Choose values that are more likely to belong to solutions
$\rightsquigarrow$ Useless for proving inconsistency of infeasible instances

- Can be learned... but this may be expensive
- Select values in the best found solution (for optimization problems)[2]

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1 Boussemart et al. *Boosting systematic search by weighting constraints*, in ECAI 2004

2 Demirovic et al. *Solution-based phase saving for CP: A value-selection heuristic to simulate local search*, in CP 2018

Overview of the talk

1 Modelling with CP

2 Generic CP Solving Algorithms

3 LAD2025: A Constraint-based Solver for the SIP

4 Conclusion

Constraint-based Solvers for the SIP

The SIP is straightforward to model with CP...

... and it's easy to add application-dependent constraints or objective functions (related to labels, for example)

... but generic CP solvers suffer from a rather long initialisation process

Constraint-based solvers dedicated to the SIP:

- Combine Branch & Propagate with dedicated data structures, heuristics, propagators, ...
- For example: Ullman[1], LV[2], ILF[3], LAD[4], SND[5], Glasgow[6], PathLAD+[7], ... and LAD2025[8]

---

1 Ullmann. *An algorithm for subgraph isomorphism*, in J. ACM 1976

2 Larrosa and Valiente. *Constraint satisfaction algorithms for graph pattern matching. Mathematical. Structures*, in Comp. Sci. 2002

3 Zampelli et al. *Solving subgraph isomorphism problems with constraint programming*, in Constraints 2010

4 Solnon. *Alldifferent-based filtering for subgraph isomorphism*, in Art. Int. 2010

5 Audemard et al. *Scoring-based neighborhood dominance for the subgraph isomorphism problem*, in CP 2014

6 McCreesh et al. *The glasgow subgraph solver: Using CP to tackle hard SIP variants*, in ICGT 2020

7 Wang et al. *PathLAD+: An Improved Exact Algorithm for Subgraph Isomorphism Problem*, in IJCAI 2023

8 Solnon: *LAD2025, a Constraint-Based Solver for the Subgraph Isomorphism Problem*, 2025 (submitted)

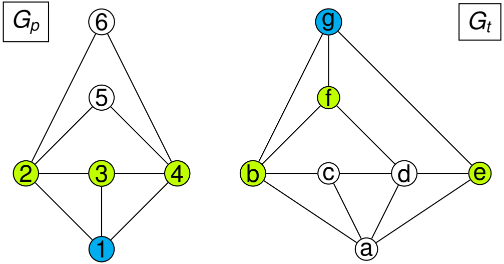
V1 of LAD[1]

⇝ Propagation of Locally All Different (LAD) constraints

Definition of LAD(x_i, u) with $i \in V_p$ and $u \in V_t$:

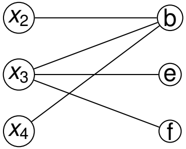
$(x_i = u) \Rightarrow (\forall j \in adj(i), x_j \in adj(u)) \wedge allDifferent(\{x_j \mid j \in adj(i)\})$

Example: Propagation of LAD when $i = 1$ and $u = g$



$D(x_2) = D(x_4) = \{a, b, d\}$
$D(x_1) = D(x_3) = D(x_5) = D(x_6) = V_t$

$(x_1 = g) \Rightarrow x_2, x_3, x_4 \in \{b, e, f\} \wedge allDifferent(x_2, x_3, x_4)$



- Propagating LAD(x_1, g) removes g from $D(x_1)$

1 Solnon. *AllDifferent-based filtering for subgraph isomorphism*, in Artificial Intelligence, 2010

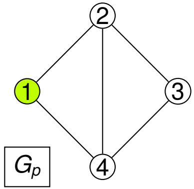
V2 of LAD = PathLAD[1]

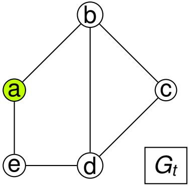
⇝ Propagation of path-based constraints

Definition of path-based constraints[2]:

- Let $p_2(i, G)$ be the number of 2-paths starting from i in G
  $\rightsquigarrow \forall i \in V_p, p_2(i, G_p) \leq p_2(x_i, G_t)$
- Let $p_2(i, j, G)$ be the number of 2-paths between i and j in G
  $\rightsquigarrow \forall i, j \in V_p, p_2(i, j, G_p) \leq p_2(x_i, x_j, G_t)$

Example:





- Propagation of $p_2(1, G_p) \leq p_2(a, G_t)$:
  $p_2(1, G_p) = 4$ and $p_2(a, G_t) = 3$
  $\rightsquigarrow$ Remove a from $D(x_1)$
- Propagation of $p_2(2, 4, G_p) \leq p_2(b, d, G_t)$ when $x_2 = b$:
  $p_2(2, 4, G_p) = 2$ and $p_2(b, d, G_t) = 1$
  $\rightsquigarrow$ Remove d from $D(x_4)$

1 Kotthoff, McCreesh, Solnon. *Portfolios of subgraph isomorphism algorithms*, in LION 2016

2 McCreesh, Prosser. *A parallel, backjumping subgraph isomorphism algorithm using supplemental graphs*, in CP 2015

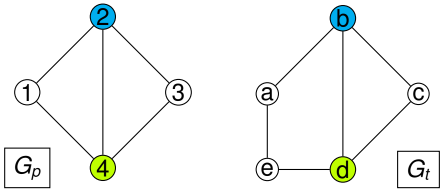
V2 of LAD = PathLAD[1]

⇝ **Propagation of path-based constraints**

Definition of path-based constraints[2]:

- Let $p_2(i, G)$ be the number of 2-paths starting from i in G
  $\rightsquigarrow \forall i \in V_p, p_2(i, G_p) \leq p_2(x_i, G_t)$
- Let $p_2(i, j, G)$ be the number of 2-paths between i and j in G
  $\rightsquigarrow \forall i, j \in V_p, p_2(i, j, G_p) \leq p_2(x_i, x_j, G_t)$

Example:



- Propagation of $p_2(1, G_p) \leq p_2(a, G_t)$:
  $p_2(1, G_p) = 4$ and $p_2(a, G_t) = 3$
  $\rightsquigarrow$ Remove a from $D(x_1)$
- Propagation of $p_2(2, 4, G_p) \leq p_2(b, d, G_t)$ when $x_2 = b$:
  $p_2(2, 4, G_p) = 2$ and $p_2(b, d, G_t) = 1$
  $\rightsquigarrow$ Remove d from $D(x_4)$

1 Kotthoff, McCreesh, Solnon. *Portfolios of subgraph isomorphism algorithms*, in LION 2016

2 McCreesh, Prosser. *A parallel, backjumping subgraph isomorphism algorithm using supplemental graphs*, in CP 2015

V3 of LAD

$\rightsquigarrow$ **Refactoring of PathLAD (mostly based on tips from Knuth[1] :-)**

Main improvements:

- Generalize the use of Sparse Sets[2] to restore states in constant time when backtracking
- Use time-stamps to wipe-out sets in constant time
- Use Tarjan instead of Kosaraju to search for strongly connected components
- Use Ford-Fulkerson instead of Hopcroft-Karp to search for covering matchings[3]
- ...

Was it worth refactoring?

1 Knuth. *The Art of Computer Programming (Fascicule 7: Constraint Satisfaction)*, 2025

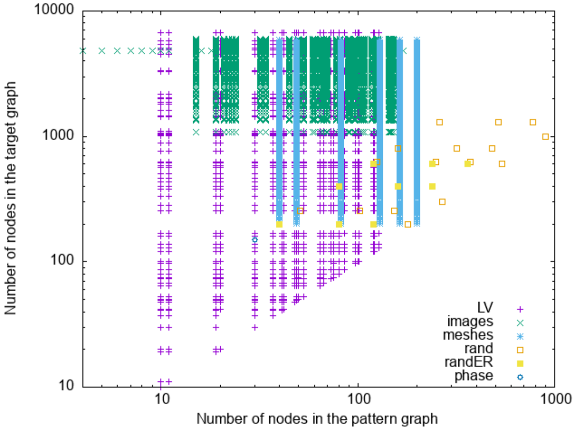
2 Le Clément, Schaus, Solnon, Lecoutre: *Sparse-Sets for Domain Implementation*, in TRICS, 2013

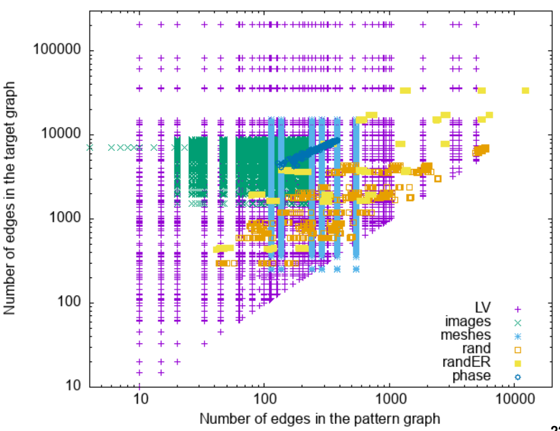
3 Gent, Miguel, Nightingale: *Generalised arc consistency for the alldifferent constraint: An empirical survey* in AI 2008

Experimental Evaluation: Benchmark Description

15,128 instances coming from 8 existing benchmarks

- Instances coming from real applications: Images and Meshes
- Random instances: randER (Erdös-Rényi, including hard instances) and rand (other models)
- Instances generated from the Stanford graph base: LV

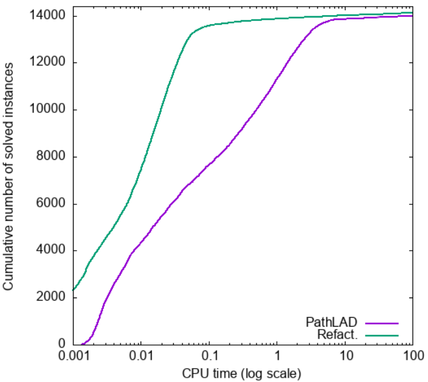


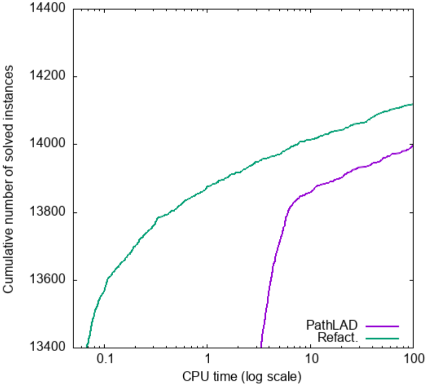


Number of instances NOT solved when the time limit is 100s (average solving time if all solved):

	LV	RandER	Meshes	Rand	Images	All benchmarks
V2 = PathLAD	222	278	51	0 (0.18s)	0 (0.86s)	551
V3 = Refactoring of V2	170	226	44	0 (0.03s)	0 (0.02s)	440

Evolution of the cumulative number of solved instances wrt time:





V4 of LAD

$\rightsquigarrow$ V3 of LAD + Propagation of Clique-based Constraints

Constraints based on maximum cliques[1]:

- Let $\chi(i, G)$ be the order of the largest clique of G that contains i
  $\rightsquigarrow \forall i \in V_p, \chi(i, G_p) \geq \chi(x_i, G_t)$
  $\rightsquigarrow$ Used to filter domains before starting the search

Constraints based on the number of k-cliques (cliques of order k), with $k \in \{3, 4, 5, 6\}$:

- Let $c(i, k, G)$ be the number of k-cliques of G that contain i
  $\rightsquigarrow \forall i \in V_p, c(i, k, G_p) \geq c(x_i, k, G_t)$
  $\rightsquigarrow$ Used to filter domains before starting the search
- Let $c(i, j, k, G)$ be the number of k-cliques of G that contain i and j
  $\rightsquigarrow \forall i, j \in V_p, c(i, j, k, G_p) \geq c(x_i, x_j, k, G_t)$
  $\rightsquigarrow$ Used to tighten LAD constraints during the search

Implementation "details":

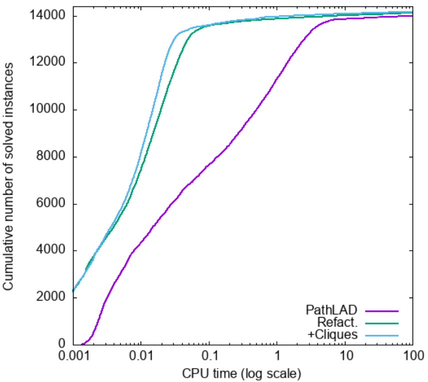
- Use efficient algorithms for computing cliques
- First compute cliques in G_p to derive bounds for the computation of cliques in G_t

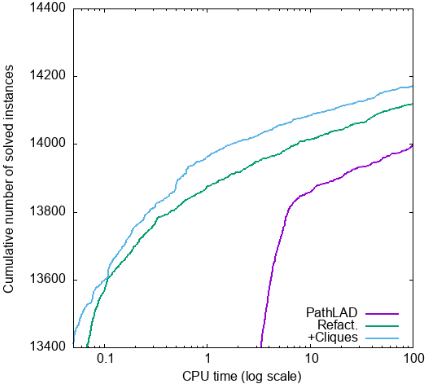
[1] Kraiczy, McCreesh. *Solving graph homomorphism and SIPs faster through clique neighbourhood constraints*, in IJCAI 2021

Number of instances NOT solved when the time limit is 100s (average solving time if all solved):

	LV	RandER	Meshes	Rand		Images		All benchmarks
V3 = Refactoring of V2	170	226	44	0	(0.03s)	0	(0.02s)	440
V4 = V3 + Cliques	142	200	45	0	(0.02s)	0	(0.01s)	387

Evolution of the cumulative number of solved instances wrt time:





V5 of LAD

$\rightsquigarrow$ **V4 of LAD + Variable Ordering Heuristic**

Variable ordering heuristic used in V4: MinDom

- Select the variable x_i that minimizes $|D(x_i)|$
  $\rightsquigarrow$ Break ties by maximizing the degree of the pattern vertex i
  $\rightsquigarrow$ Break further ties randomly

Variable ordering heuristic used in V5: MinDom/wdeg[1]

- Select the variable x_i that minimizes

$$\frac{|D(x_i)|}{\sum_{c \in C(x_i)} w(c)}$$

  where

  - $C(x_i)$ = set of constraints that involve x_i and at least one unassigned variable
  - $w(c)$ = number of times the propagation of c has raised a failure

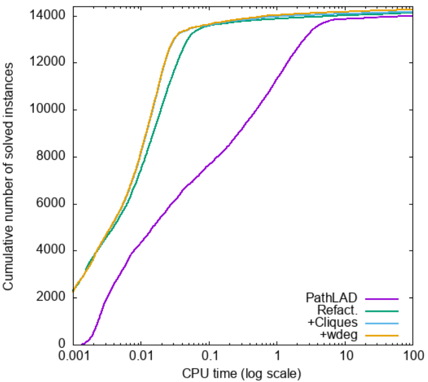
$\rightsquigarrow$ Dynamically identify critical variables that must be assigned first

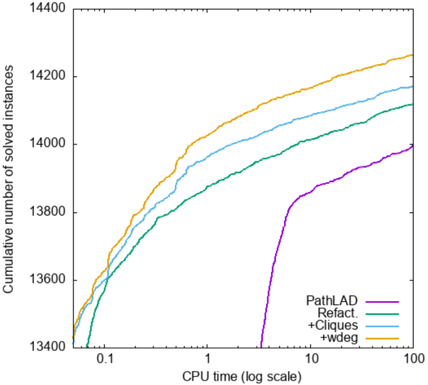
[1] Boussemart et al. *Boosting systematic search by weighting constraints*, in ECAI 2004

Number of instances NOT solved when the time limit is 100s (average solving time if all solved):

	LV	RandER	Meshes	Rand	Images	All benchmarks
V4 = V3 + Cliques	142	200	45	0 (0.02s)	0 (0.01s)	387
V5 = V4 + minDom/wdeg	105	187	1	0 (0.02s)	0 (0.01s)	293

Evolution of the cumulative number of solved instances wrt time:





V6 of LAD

⇝ V5 of LAD + Restarts and Randomized Value Ordering Heuristic

Restarts[1]:

- Restart the search when the number of failures reaches a cutoff limit
- Geometric progression of the cutoff limit from one restart to the next one

⇝ Avoids heavy tails when combined with randomized ordering heuristics

NoGood recording[2]:

- After each restart, add a constraint to forbid the exploration of nodes already explored

Value Ordering Heuristic:

- Degree-biased heuristic[3]: The probability to choose a target vertex is proportional to its degree
- Combined with saving: Give priority to values used in the largest assignment built so far

---

1 Gomes, Selman, Kautz: *Boosting combinatorial search through randomization*, in AAAI 1998

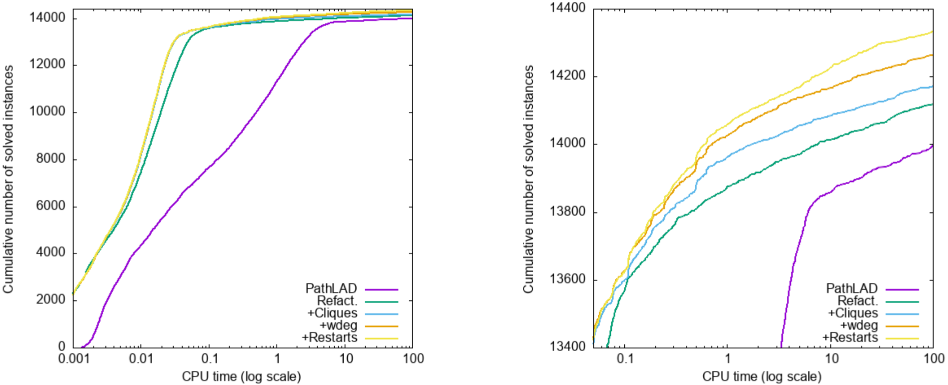
2 Lee, Schulte, Zhu: *Increasing nogoods in restart-based search*, in AAAI 2016

3 Archibald et al: *Sequential and parallel solution-biased search for subgraph algorithms*, in CPAIOR 2019

Number of instances NOT solved when the time limit is 100s (average solving time if all solved):

	LV	RandER	Meshes	Rand	Images	All benchmarks
V5 = V4 + minDom/wdeg	105	187	1	0 (0.02s)	0 (0.01s)	293
V6 = V5 + Restarts	91	133	1	0 (0.02s)	0 (0.01s)	225

Evolution of the cumulative number of solved instances wrt time:



V7 of LAD = LAD2025

⇝ V6 of LAD + Per Instance Propagation Selection

Two complementary propagation algorithms for the subgraph isomorphism problem:

- DC of LAD constraints: Drastically filters domains, but very expensive on dense graphs
- FC of basic edge constraints: Very fast, but filters less values

The "best" propagation algorithm depends on the instance to solve!

Per instance algorithm selection[1]:

Given a portfolio of solvers with complementary performance:

- Learn a model for selecting the best performing solver for each instance
- Use this model to select a solver at run time

Simple rule for selecting the propagation algorithm:

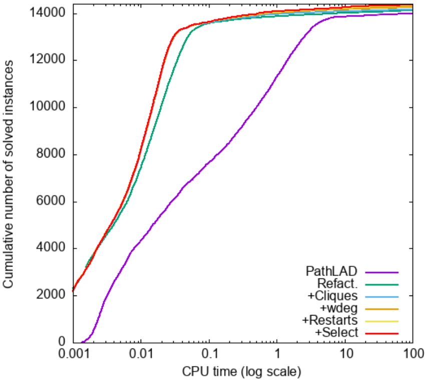
- If the density of G_p and G_t exceeds 0.15: Select FC of edge constraints
- Otherwise: Select DC of LAD constraints

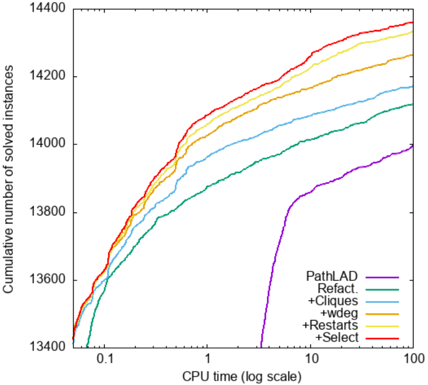
[1] Kotthoff, McCreesh, Solnon: *Portfolios of subgraph isomorphism algorithms*, in LION 2016

Number of instances NOT solved when the time limit is 100s (average solving time if all solved):

	LV	RandER	Meshes	Rand		Images		All benchmarks
V6 = V5 + Restarts	91	133	1	0	(0.02s)	0	(0.01s)	225
V7 = V6 + Select	91	104	1	0	(0.02s)	0	(0.01s)	196

Evolution of the cumulative number of solved instances wrt time:





Experimental comparison with state-of-the-art approaches

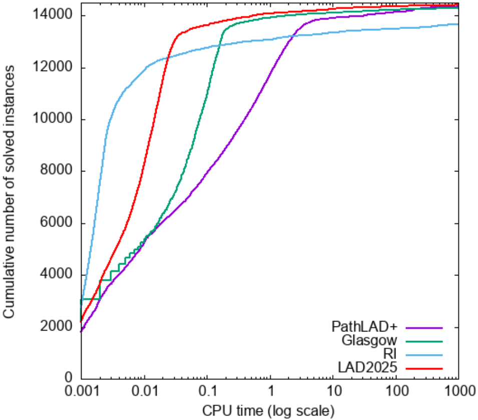
Number of instances NOT solved when the time limit is 1000s (average solving time if all solved):

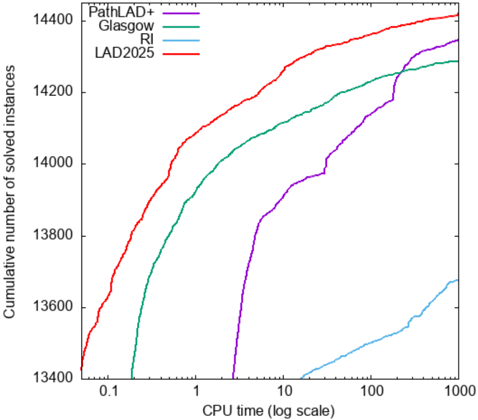
	LV	RandER	Meshes	Rand		Images		All benchmarks
RI[1]	356	180	324	20	-	0	(0.003s)	880
Glasgow[2]	152	86	32	0	(0.042s)	0	(0.080s)	270
PathLAD+[3]	112	89	11	0	(0.198s)	0	(0.842s)	212
LAD2025[4]	72	68	1	0	(0.024s)	0	(0.014s)	141

- RI is the fastest approach on Images, but it is much less successful on all other benchmarks
- LAD2025 is the most successful solver on all other benchmarks

1 Bonnici, Giugno. *On the variable ordering in subgraph isomorphism algorithms*, in IEEE Trans. Bioinform., 2017
2 Kraiczy, McCreesh. *Solving graph homomorphism and SIPs faster through clique neighbourhood constraints*, in IJCAI 2021
3 Wang, Jin, Cai, Lin. *PathLAD+: An Improved Exact Algorithm for Subgraph Isomorphism Problem*, in IJCAI 2023
4 Solnon: *LAD2025, a Constraint-Based Solver for the Subgraph Isomorphism Problem*, 2025 (submitted)

Cumulative Number of Solved Instances wrt Time





- RI is the most successful solver when the time limit is smaller than 0.03s
  ⇝ But it solves 739 less instances than LAD2025 when the time limit is 1000s
- LAD2025 is the most successful solver when the time limit is greater than 0.03s

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4 Conclusion

Conclusion

Modelling and solving the subgraph isomorphism problem with CP

- Very nice and straightforward model
  $\rightsquigarrow$ Application-dependent constraints and/or objective functions may be easily added
- Generic CP solvers are able to solve medium-size instances...
  ...But they are not as efficient as state-of-the-art dedicated approaches

Constraint-based approaches for solving the subgraph isomorphism problem

- Combine classical ingredients of CP solvers
  $\rightsquigarrow$ Constraint propagation, Ordering heuristics, Random restarts, NoGood recording, ...
- With efficient data structures
  $\rightsquigarrow$ Sparse sets, Time stamps, ...
- And dedicated graph invariants
  $\rightsquigarrow$ 2-paths, max cliques, k-cliques

Some further work

- Extension to directed and labelled graphs, and to the induced subgraph isomorphism problem